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LETTER TO THE EDITOR

Universality of node-avoiding and path-avoiding Levy flights

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Abstract. The real space renormalisation group theory is applied to the self-avoiding Levy flight problem with Levy exponent μ on a two-dimensional square lattice. A finite-lattice renormalisation transformation is used to derive the critical behaviour exhibited by two classes of self-avoiding Levy processes: the node-avoiding Levy flight (NALF) and the path-avoiding Levy flight (PALF). It is found that the NALF and the PALF belong to the same universality class.

Random walks in which the step length is a random variable with an infinite mean-squared displacement per step are called Levy flights (Levy 1937, Montroll and West 1979, Mandelbrot 1977). The Levy flight process is characterised by the Levy exponent μ which parametrises the single-step probability $\tau_\mu(l)$ of taking a step of length l. Asymptotically, $\tau_\mu(l) \sim l^{-(\mu+d)}$ in d dimensions. If $0<\mu<2$, then the second moment $\langle l^2\rangle$ is infinite and the corresponding Levy process is 'superdiffusive'. If $\mu \geqslant 2$, then $\langle l^2\rangle$ is finite and thus, via the central limit theorem, leads to ordinary diffusive behaviour. The self-similar structure of the Levy flight has been characterised by a fractal dimension $D=\mu$ for $\mu<2$ (Mandelbrot 1977, Hughes *et al* 1981, Seshadri and West 1982). The connection between the Levy flight and the long-range interacting spherical model of critical phenomena has been noted (Joyce 1972, Hioe 1984).

There exist two distinct types of self-avoiding interactions that can be imposed on the Levy flight process (Halley and Nakanishi 1985). For a Levy flight on a lattice, the unequal step lengths provide the opportunity for a Levy flight configuration to contain path intersections in addition to the ordinary node intersections. The path-avoiding Levy flight (PALF) is defined as a Levy flight that must avoid any part (nodes and bonds) of its entire path. Path-avoidance is a stronger constraint than node-avoidance and thus PALF represent a subset of the class of node-avoiding Levy flights (NALF). PALF would be very difficult to simulate on the computer because the path-avoiding constraint represents a long-range (non-Markovian) interaction that requires the memorisation of a set of paths, rather than a set of points.

Recently, there has been considerable interest in the node-avoiding Levy flight (NALF). The effect of a node-avoiding constraint on the Levy process has been studied via numerical simulations in one (Grassberger 1985) and two (Halley and Nakanishi 1985) dimensions. The isomorphism between the NALF and a zero-component spin model with long-range interactions (Fisher *et al* 1972) has been established (Halley and Nakanishi 1985). A direct renormalisation group theory of the NALF, represented as a geometrical equilibrium statistical mechanical model, has been developed and used to derive the critical exponents and the end-to-end distance probability function

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through first order in $\varepsilon = 2\mu - d$ (Prentis 1985). The NALF represents the first concrete realisation of a many-body system that can assume a continuum of values of ε near zero.

In this letter, we develop a real space renormalisation theory of the self-avoiding Levy flight to understand the critical behaviour exhibited by both the NALF and the PALF. The path-avoiding constraint, which is difficult to formulate analytically or simulate numerically, is readily incorporated into the framework of a real space renormalisation theory. We study the connection between the NALF and the PALF and establish the universality between these two classes of self-avoiding Levy flights.

We consider an equilibrium statistical mechanical model of self-avoiding Levy flights on a two-dimensional square lattice. In addition to the fugacity weight K associated with each step of a Levy flight configuration, there exists the normalised weight $\tau_\mu(l)$ for each step of length l. This normalised weight is taken to be the single-step probability function that defines the Levy process on a lattice (Hughes *et al* 1981, Montroll and West 1979, Halley and Nakanishi 1985):

$$\tau_\mu(l) = C \sum_n a^{-n} \delta_{l,b^n}. \qquad (1)$$

In this expression, a and b are real numbers greater than unity and C is a normalisation constant. This probability function defines a lattice Levy flight of order μ where the Levy exponent μ is (Hughes *et al* 1981):

$$\mu = \ln a / \ln b. \qquad (2)$$

To construct the renormalisation transformation, we partition the lattice into 2×2 cells (length rescaling factor $s = 2$) and consider an equilibrium ensemble of self-avoiding Levy flights on a finite lattice defined by one 2×2 cell. Each step of a Levy flight configuration is assigned the weight $K\tau_\mu(l)$. The Levy parameter b in equation (1) is assigned the value $b = \sqrt{2}$. This allows steps of length $l = 1, \sqrt{2}, 2$ and $\sqrt{8}$ to exist on the finite lattice. This range of step lengths determines the normalisation constant C in equation (1) and thus uniquely defines the single-step probability function $\tau_\mu(l)$ on the finite lattice. In the renormalised system, the renormalised fugacity K' is obtained by a connectivity rule (Stanley *et al* 1982). The set of all self-avoiding Levy flights that span the cell by beginning at the lower left vertex and exiting at the upper left vertex map onto a single vertical renormalised step of fugacity K'. Figure 1 illustrates this finite lattice renormalisation scheme for both a NALF and a PALF configuration.

The critical point $K_{c\mu}$ and the critical exponent ν_μ of the self-avoiding Levy flight of order μ are obtained in the usual way (Stanley *et al* 1982) from the fixed point K_μ^*

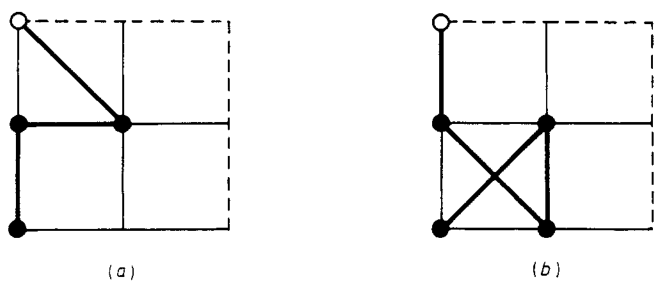

Figure 1. Examples of (a) path-avoiding Levy flight (PALF) and (b) node-avoiding Levy flight (NALF) that span the finite lattice and map onto a single renormalised step.

and the eigenvalue λ_μ of the renormalisation transformation $K'_\mu(K)$:

$$K_{c\mu} = K^*_\mu \qquad \text{where } K'_\mu(K^*_\mu) = K^*_\mu$$
$$\nu_\mu = \frac{\ln 2}{\ln \lambda_\mu} \qquad \text{where } \lambda_\mu = \left.\frac{\mathrm{d}K'_\mu}{\mathrm{d}K}\right|_{K=K^*_\mu}. \tag{3}$$

Interpretation of the critical exponent ν_μ as a measure of the average size of the self-avoiding Levy flight must be done with caution. This is due to the existence of infinite moments which are the trademark of the Levy flight process. More precisely, the exponent ν characterises the scaling function describing the end-to-end distance R correlation function (Prentis 1985). The generalised moments of order m of an N-step Levy flight are finite $\langle R^m \rangle \sim N^{m\nu}$ if $m < \mu$ and infinite otherwise.

The critical exponents ν_μ obtained from this finite lattice renormalisation are displayed in figure 2. In this approximation, the exponents characterising the NALF and the PALF differ at most by 2% and are indistinguishable on the graph of figure 2.

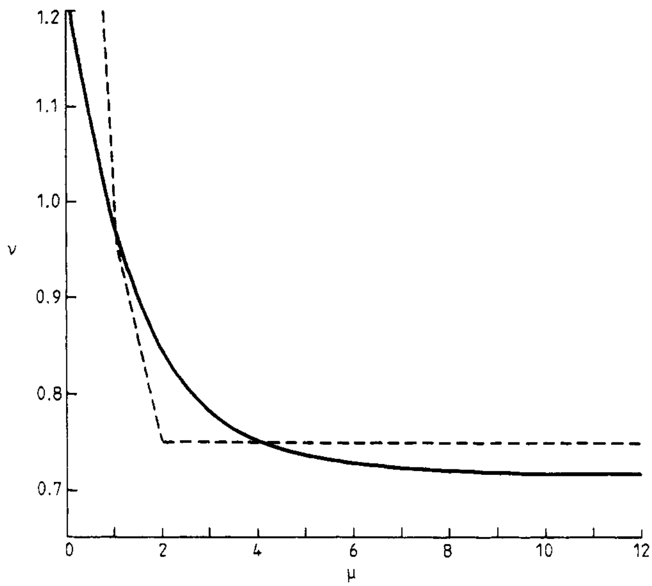

Figure 2. Critical exponent ν characterising the self-avoiding Levy flight of order μ. The full curve is obtained from the finite-lattice renormalisation theory and describes both the NALF and the PALF. The broken curve is the prediction of a Flory theory of the NALF.

Also shown in figure 2 is the prediction of the Flory theory of the NALF (Grassberger 1985). We have also implemented our renormalisation program for self-avoiding Levy flights on a 3×3 lattice and find similar behaviour. Although ν_μ for the NALF and the PALF are almost identical in the finite lattice approximation, this cannot be taken as evidence of their universality.

In order to address the question of universality, we introduce an additional parameter that will provide a mechanism to continuously connect the NALF and PALF problems. The natural parameter to accomplish this is a path interaction weight v with magnitude $0 \leq v \leq 1$ that is to be assigned to all path intersections in a Levy flight configuration. The special case $v = 0$ $(v = 1)$ represents the PALF (NALF) class of self-avoiding Levy flights. Formally, this interaction parameter is the path-avoiding analogue of the Domb-Joyce parameter that characterises the cluster expansion of the node-avoiding random walk (polymer) problem on a lattice (Domb and Joyce 1972). A similar parameter was utilised in the real space renormalisation approach to understand the crossover between random and node-avoiding random walks (Family and Gould 1984).

The renormalised path interaction parameter v' is the weight associated with a path intersection in the renormalised system. In the finite lattice approximation, it is calculated from the set of all self-avoiding Levy flight configurations that span the cell between the two pairs of diagonal entrance and exit points. Each of these configurations represent a path-intersecting portion of an infinite lattice Levy flight. This set of Levy flight configurations will rescale onto a renormalised Levy flight that contains a path intersection. The renormalisation of the path-interaction weight v is illustrated in figure 3.

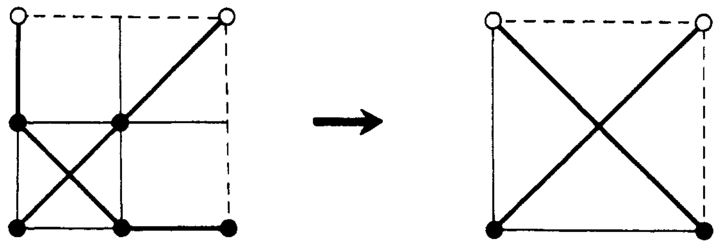

Figure 3. Example of a Levy flight configuration on the finite lattice that renormalises into a path-intersecting Levy flight.

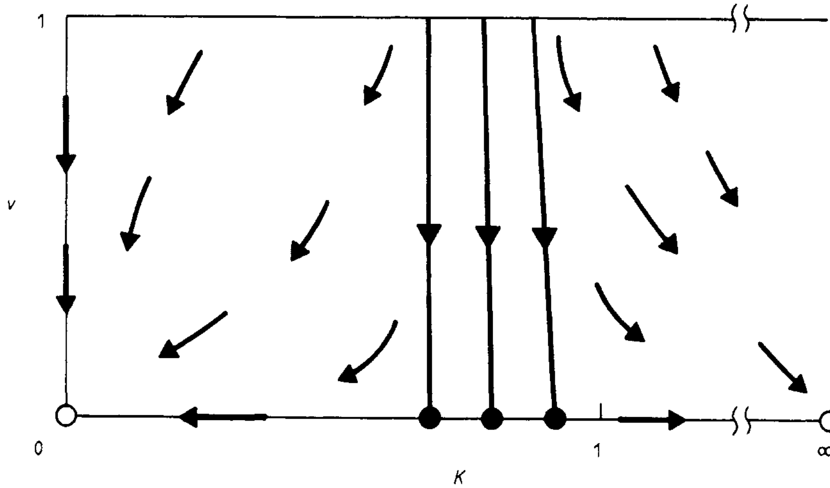

Figure 4. Phase diagram representation of the renormalisation group flows. For each value of the Levy exponent μ, there exists a non-trivial PALF ($v=0$) fixed point K_μ^* (●) whose critical surface intersects the NALF ($v=1$) axis. The fixed points ($K_\infty^* < K_2^* < K_1^*$) and critical surfaces for $\mu = 1, 2$ and ∞ are shown in the diagram. The trivial PALF fixed points (○) at $K=0$ and $K=\infty$ represent attractors for the domains to the left and right, respectively, of a given critical surface.

This finite lattice renormalisation program yields a two-dimensional renormalisation group transformation that assumes the form:

$$K'_\mu \tau_\mu(1') = K\tau_\mu(2) + K^2(\tau_\mu^2(1) + \tau_\mu^2(\sqrt{2})) + 4K^3\tau_\mu^2(1)\tau_\mu(\sqrt{2})$$
$$+ K^4(\tau_\mu^4(1) + (2+v)\tau_\mu^2(1)\tau_\mu^2(\sqrt{2})) \tag{4}$$
$$v'(K'_\mu)^2\tau_\mu^2(\sqrt{2}') = v[K^2\tau_\mu^2(\sqrt{8}) + K^5\tau_\mu^2(1)\tau_\mu^3(\sqrt{2})]. \tag{5}$$

These equations define a two-dimensional renormalisation group mapping: $K'_\mu(K, v)$ and $v'_\mu(K, v)$. The renormalisation group flows characterising this map are represented in the phase diagram of figure 4. Upon repeated renormalisation, the path interaction parameter renormalises to zero. This is a manifestation of the fact that path intersections create loops which eventually become undone through renormalisation. This is a statement that path intersections in the Levy flight process are irrelevant to the critical behaviour. A similar phenomenon has been observed in the study which demonstrates the irrelevance of loops in the lattice trail problem (Family 1982). A feature of more significance is that for each value of μ, there exists a non-trivial PALF ($v = 0$) fixed point K_μ^* whose critical surface (curve) intersects the NALF ($v = 1$) axis. This demonstrates that the NALF and the PALF problems belong to the same universality class.

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